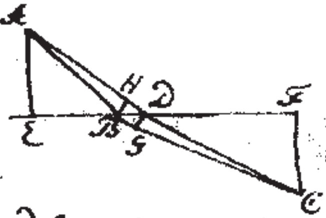
Invenire curvam celerrimi descensus in medio resistente positis actionibus gravitatis parallelis.

Sit $ABC$ portio infinite parva curvae quaesitae. Ducatur horizontalis $EBF$ ita ut in illam demissae normales $AE$, $CF$ fiant aequales, assumtoque in $EF$ puncto $D$ ipsi $B$ proximo, ducantur $AD$, $DC$, centrisque $A$ et $C$ fiant arcus $BH$, $DG$, sit $AE = FC = dy$, $EB = dx$, $AB = ds$, unde $BF = dx + ddx$, $BC = ds + dds$. Dicatur $BD = o$, eritque $HD = \frac{odx}{ds}$, et $BG = \frac{odx + oddx}{ds + dds} = \frac{odx}{ds} + \frac{oddx}{ds} - \frac{odx\,dds}{ds^2}$ continue dividendo numeratorem per denominatorem, adeoque $AD = ds + \frac{odx}{ds}$, et $DC = ds + dds - \frac{odx}{ds} - \frac{oddx}{ds} + \frac{odx\,dds}{ds^2}$.




Velocitas in $A$ qua percurruntur $AB$ et $AD$ sit $v$, velocitas in $B$ qua percurritur $BC$, $v + dv$, si nulla esset medii resistentia velocitas in $B$ aequalis esset velocitati in $D$. Sed quoniam resistentia quae dicatur $r$ diutius agit contra descensum per $AD$ quam contra descensum per $AB$, existente excessu temporis $\frac{HD}{v}$ sive $\frac{odx}{vds}$, quo tempore generat decrementum velocitatis $\frac{rdx}{vds}$, erit velocitas in $D$ qua percurritur $DC = v + dv - \frac{rodx}{vds}$. Tempus itaque per $AB$ erit $\frac{ds}{v}$, per $BC$ $\frac{ds + dds}{v + dv}$

$$= \frac{ds}{v} - \frac{ds\,dv}{v^2} + \frac{ds\,dv^2}{v^3} + \frac{dds}{v} - \frac{dds\,dv}{v^2},$$

actualem instituendo divisionem. Tempus per $AD = \frac{ds}{v} + \frac{odx}{vds}$, et per

$$DC = \frac{ds + dds - \frac{odx}{ds} - \frac{oddx}{ds} + \frac{odx\,dds}{ds^2}}{v + dv - \frac{rodx}{vds}}$$

$$= \left\{ \begin{array}{l} \frac{ds}{v} + \frac{dds}{v} - \frac{dds\,dv}{vv} \\ \quad - \frac{ds\,dv}{vv} + \frac{ds\,dv^2}{v^3} \\ \quad - \frac{odx}{vds} + \frac{odx\,dv}{v^2 ds} \\ \quad\quad - \frac{oddx}{vds} \\ \quad\quad + \frac{odx\,dds}{vds^2} \\ \quad\quad + \frac{rodx}{v^3} \end{array} \right\}$$

Statuatur jam, per naturam maximi et minimi, summa temporum per $AB$ et $BC$ aequalis summae temporum per $AD$ et $DC$, unde deletis quae se mutuo

tollunt

tollunt, et dividendo quae restant per $\frac{odx}{vds}$ invenitur $\frac{dds}{ds}$ $-\frac{ddx}{dx}+\frac{rds}{v^2}+\frac{dv}{v}=0$. Haec aequatio comparata cum aequatione $pdy-rds=vdv$, quae in omni descensu per medium resistens obtinet ubi vis gravitatis est $p$ curvam quaesitam determinat.

quoniam per aequationem posteriorem est $\frac{rds}{v^2}+\frac{dv}{v}=\frac{pdy}{v^2}$, aequatio prior facta substitutione, evadet $\frac{dds}{ds}-\frac{ddx}{dx}+\frac{pdy}{v^2}=0$, quae facta reductione ope aequationum $ds^2=dx^2+dy^2$, et $dsdds=dxddx$, fit $v^2=\frac{pds^2dx}{dyddx}=\frac{pdx^2ds}{dydds}$. Quare per hanc aequationem cum aequatione $pdy-rds=vdv$ comparatam Problema propositum solvitur. Scilicet exterminando $v$, quod fit differentiando aequationem $v^2=\frac{pds^2dx}{dyddx}=\frac{pdx^2ds}{dydds}$, habetur $2rdsdy+3pd$ $-\frac{pdx^2dsd^3s}{dds^2}=0$, ubi pro $r$ scribendo ejus valorem in $v$ sive $\sqrt{\frac{pdx^2ds}{dydds}}$, habetur aequatio curvae.

Ut calculus evadat facilior, ponatur $ds=qdy$, eritque $dx$ $\overline{q^2-1}.dy^2$, $dds=dqdy$, $d^3s=ddqdy$, et factis substitutionibus a quationes inventae $v^2=\frac{pdx^2ds}{dydds}$, et $2rdsdy+3pdx^2=\frac{pd}{dd}$ transibunt in sequentes $v^2=\frac{pqdy.\overline{q^2-1}}{dq}$, et $\frac{2rdq}{p.\overline{q^2-1}}+\frac{3dq}{q}=$

Exemplum. Sit resistentia proportionalis dignitati velocitatis cujus index est $2n$ sive $r=\frac{cp^nq^n.\overline{q^2-1}^n.dy^n}{dq^n}$. Substituto hoc valore ipsius $r$ in aequatione novissima, ipsaque ut decet disposita, habetur $2ncp^{n-1}dy^n.\frac{\overline{q^2-1}^{n-1}}{q^{2n}}dq=\frac{ndq^{n-1}}{q^{3n+1}}.qdd$ $-3dq^2$. Sumtisque integralibus $2ncp^{n-1}dy^n.\int\frac{\overline{q^2-1}^{n-1}}{q^{2n}}$ $=\frac{dq^n}{q^{3n}}$, extractaque radice cujus exponens est $n$ habetur $\overline{\frac{2nc}{p}}^{\frac{1}{n}}pdy\left(\int\frac{\overline{q^2-1}^{n-1}}{q^{2n}}dq\right)^{\frac{1}{n}}=\frac{dq}{q^3}$, et pro $\left(\int\frac{\overline{q^2-1}^{n-1}dq}{q^{2n}}\right)^{\frac{1}{n}}$ brevitatis causa scribendo $\frac{1}{Q}$, $dy=\overline{\frac{p}{2nc}}^{\frac{1}{n}}\frac{Qdq}{pq^3}$, unde

$Qds$

$$ds = q\,dy = \left(\frac{p}{2nc}\right)^{\frac{1}{n}} \frac{2\,dq}{pq^2}, \text{ et } dx = dy\sqrt{q^2-1} = \left(\frac{p}{2nc}\right)^{\frac{1}{n}} \frac{2\,dq.\sqrt{q^2-1}}{pq^3}$$

quae aequatio offerebit naturam curvae.