ESSAYS

ON SEVERAL

Curious and Uſeful SUBJECTS,

In SPECULATIVE and MIX'D

MATHEMATICKS.

Illuſtrated by a Variety of EXAMPLES.

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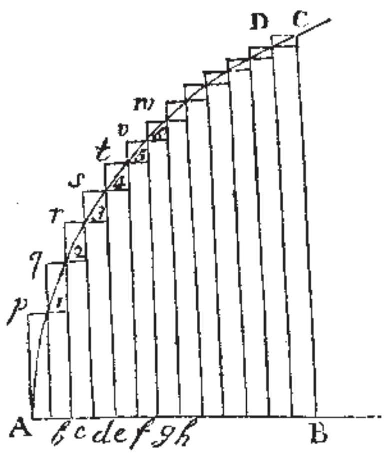
This I ſhall endeavour to ſhew by the following Inſtance; wherein A C, being ſuppoſed a Curve, whoſe Equation is $y = z^n$ (AB being equal z, and CB equal y) the Area ABC is required.



Let AB be divided into any Number, x, of equal Parts, as Ab, bc, cd, &c. and from the Points of Diviſion let Perpendiculars be raiſed, cutting the Curve in the Points, 1, 2, 3, &c. and having made p1, q2, r3, s4, &c. parallel to AB, let the Baſe Ab, bc, cd, &c. of each of the Rectangles pb, qc, rd, &c. be repreſented by d: Then b1, c2, d3, &c. the Heights of thoſe Rectangles, being Ordinates to the Curve, will be d^n, $\overline{2d}|^n$, $\overline{3d}|^n$, &c. reſpectively, each of which ∵ being multiplied by d, the common Baſe, and the Sum of all the Products taken, will give d into $d^n + \overline{2d}|^n + \overline{3d}|^n \ldots \overline{xd}|^n$, (= A$p1q2r$, &c. CBA) for the Area of the whole circumſcribing Polygon; and this Series, according to the abovefaid Theorem (*Cor.* III.) is equal to d^{n+1} *in*, $\frac{nx^{n+1}}{n+1} + \frac{x^n}{2}$ &c. $= \frac{\overline{dx}|^{n+1}}{n+1} + \frac{d \times \overline{ax}|^n}{2}$, &c. or, becauſe $dx = z$, it will be $= \frac{z^{n+1}}{n+1} + \frac{dx^n}{2}$, &c. Now, if from this the Difference of the Inſcribed and circumſcribed Polygons, or the

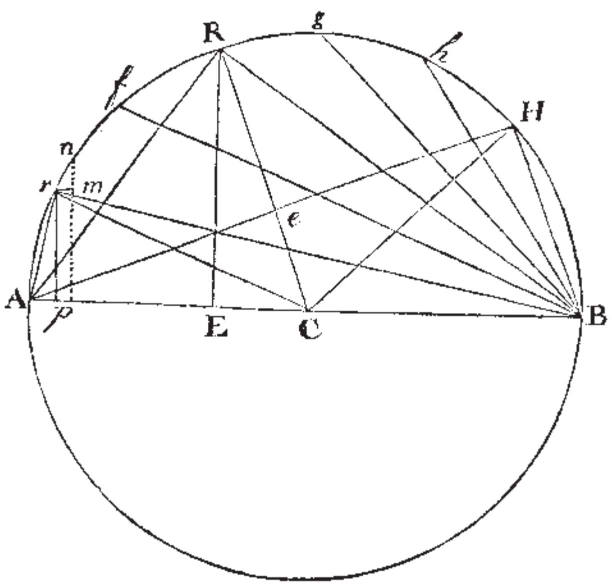
Rectangle BD $= dz^n$ be taken, there will remain $\frac{z^{n+1}}{n+1} - \frac{dzn}{2}$, for the Area of the inſcribed Polygon. Hence, it is manifeſt, that, let d be what it will, the inſcribed Polygon can never be ſo great, nor the circumſcribed ſo ſmall, as $\frac{z^{n+1}}{n+1}$ $\left(= \frac{AB \times BC}{n+1}\right)$: And therefore this Expreſſion muſt be *accurately* equal to the required Curvilinear Area ACB.

Of Angular Sections, and ſome remarkable Properties of the Circle.

PROPOSITION I.

The Radius AC, *and the Chord, Sine, or Co-ſine of an Arc, as* Ar, *being given; to find the Chord, Sine, or Co-ſine of* AR $= m \times$ Ar, *a Multiple of that Arc.*

LET RH be taken = AR, and the whole Arc AH be divided into as many equal Parts, Ar, rf, &c. as there be Units in $2m$; and the Chords Br, Bf, &c. are drawn, as alſo the Radii Cr, CR, CH, and the Perpendiculars rp, RE; calling AC, 1; Br, y; Cp, x; CE, X; rp, u; RE, U; Ar^2, z; and AH^2 = Z: Then, becauſe any one of thoſe Chords, as Bf, is to Br + BR, the Sum of the 2 next it, as BC to Br, by a known Property of the Circle, we ſhall have $y \times$ Bf = Br + BR, or $y \times$ Bf — Br = BR; and for the very ſame Reaſon, $y \times$ BR — Bf = Bg, and $y \times$ Bg — BR = Bb, &c. &c. Hence, it appears, that the Values of the Chords Bf, BR, &c. (which to a Radius equal A B, will be Co-fines of the Angles A B f,



A B R, &c.) may be readily had one after another, by taking continually the Product of the laft by y, *minus* the laft but one, for the next following: And thus are had,

$$y^2 - 2 = Bf,$$
$$y^3 - 3y = BR,$$
$$y^4 - 4y^2 + 2 = Bg,$$
$$y^5 - 5y^3 + 5y = Bh,$$
$$y^6 - 6y^4 + 9y^2 - 2 = BH,$$
&c. &c.

And generally, fuppofing $\acute{A}y^{n-1} - \acute{B}y^{n-3} + \acute{C}y^{n-5}$, &c. to denote any one Chord of the foregoing Order, and Ay

—

$-By^{n-2} + Cy^{n-4}$, &c. the next to it; then the Chord next following thefe will be $Ay^{n+1} - By^{n-1} + Cy^{n-3}$ &c. $- \acute{A}y^{n-1} + \acute{B}y^{n-3} - \acute{C}y^{n-5}$, &c. $= Ay^{n+1}$ $\left.\begin{matrix}B\\ \acute{A}\end{matrix}\right\} y^{n-1} + \left.\begin{matrix}C\\ \acute{B}\end{matrix}\right\} y^{n-3} + \left.\begin{matrix}D\\ \acute{C}\end{matrix}\right\} y^{n-5}$, &c. From which (by the Method of Increments foregoing) A will come out $= 1$, $B = n$, $C = n \times \frac{n-3}{2}$, $D = n \times \frac{n-4}{2} \times \frac{n-5}{3}$, $E = n \times \frac{n-5}{3} \times \frac{n-6}{3} \times \frac{n-7}{4}$, &c. and confequently $Ay^n - By^{\frac{n-2}{2}} + Cy^{n-4}$, &c. $= y^n - ny^{n-2} + n \times \frac{n-3}{2} y^{n-4} - n \times^{n-4} \times \frac{n-5}{2} y^{n-6} + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} y^{n-8}$, &c. wherein if n be taken equal to the given Number m, it will become $y^m - my^{m-2} + m \times \frac{m-3}{2} y^{m-4}$, &c. equal B R; but if n be equal $2m$, then it will be $y^n - ny^{n-2} + n \times \frac{n-3}{2} y^{n-4}$, &c. equal B H; where the Series are to be continued till the Exponents become negative. Hence, becaufe B f is equal $2x$, and the Arc A H $= m \times$ A f, it follows, that the Chord H B will be $= \overline{2x}|^m - m \times \overline{2x}|^{m-2} + m \times \frac{m-3}{2} \times \overline{2x}|^{m-4}$, &c. and therefore, X (= C E) the required Co-fine being equal $\frac{1}{2}$ H B, we have $X = \frac{\overline{2x}|^m}{2} - \frac{m}{2} \times \overline{2x}|^{m-2} + \frac{m}{2} \times \frac{m-3}{2} \times \overline{2x}|^{m-4} - \frac{m}{2} \times \frac{m-4}{2} \times \frac{m-5}{3} \times \overline{2x}|^{m-6} + \frac{m}{2} \times \frac{m-5}{2} \times \frac{m-6}{3} \times \frac{m-7}{4} \times \overline{2x}|^{m-8}$ &c. fhewing the Relation of the Co-fines; from whence

U

U ($=\sqrt{1-X^2}$) comes out $=\sqrt{1-xx}$, *in*, $\overline{2x}|^{m-1} - \frac{m-2}{1}\times\overline{2x}|^{m-3} + \frac{m-3}{1}\times\frac{m-4}{2}\times\overline{2x}|^{m-5} - \frac{m-4}{1}\times\frac{m-5}{2}\times\frac{m-6}{3}\times\overline{2x}|^{m-7}$, &c. Furthermore, because $\frac{BH}{2}$ is equal CE, $= \frac{y^n - ny^{n-2} + n\times\frac{n-3}{2}y^{n-4},\ \&c.}{2}$, A E will be equal to $\frac{-y^n + ny^{n-2} - n\times\frac{n-3}{2}y^{n-4},\ \&c.}{2} + 1$; and therefore A E × A B equal to $-y^n + ny^{n-2} - n\times\frac{n-3}{2}y^{n-4}$ &c. $+ 2 = Z$, where, if instead of yy, its Equal $-z+4$ ($AB^2 - Ar^2$) be substituted, it will become $Z = \pm z^{\frac{n}{2}} \mp nz^{\frac{n-2}{2}} \pm n\times\frac{n-3}{2}z^{\frac{n-4}{2}}$, &c. equal $\pm z^m \mp 2mz^{m-1} \pm 2m\times\frac{2m-3}{2}\times z^{m-2} \mp 2m\times\frac{2m-4}{3}\times\frac{2m-5}{3}\times z^{m-3} \pm 2m\times\frac{2m-5}{2}\times\frac{2m-6}{3}\times\frac{2m-7}{4}\times z^{m-4}$, &c. continued to as many Terms as there are Units in m. *Q. E. I.*

Otherwise,

Let the Lines rp, R E, be considered in a flowing State, and (mn) as equal to $\dot{x}$; then we shall have $\sqrt{1-xx}$ (pr) : 1 (Cr) :: $\dot{x}$: $\frac{\dot{x}}{\sqrt{1-xx}}$ equal rn; and this being the Fluxion of the Arc Ar, that of A R (equal $m\times$ Ar) will be $\frac{m\dot{x}}{\sqrt{1-xx}}$; which, for the very same Reason

that

that $\frac{\dot{x}}{\sqrt{1-xx}}$ is the Fluxion of the Ar, must be equal to $\frac{\dot{X}}{\sqrt{1-X^2}}$: Whence, equally multiplying the two Denominators by $\sqrt{-1}$, we get $\frac{m\dot{x}}{\sqrt{xx-1}} = \frac{\dot{X}}{\sqrt{X^2-1}}$; where, taking the Fluent on each Side, there comes out, either, Log. $X+\sqrt{X^2-1} = m\times$ Log. $x+\sqrt{xx-1}$, or, Log. $X-\sqrt{XX-1} = m$ Log. $x-\sqrt{xx-1}$; wherefore, $X+\sqrt{X^2-1}$ and $\overline{x+\sqrt{xx-1}}|^m$, as also, $X-\sqrt{X^2-1}$ and $\overline{x-\sqrt{xx-1}}|^m$, the Numbers corresponding to those Logarithms must be equal: Hence, by adding together the two Equations, we have $2X = \overline{x+\sqrt{xx-1}}|^m + \overline{x+\sqrt{xx-1}}|^m$, and by taking their Difference, $2\sqrt{X^2-1} = \overline{x+\sqrt{xx-1}}|^m - \overline{x-\sqrt{xx-1}}|^m$; from whence, by expanding the latter Part of each of the Equations into Series, and dividing the whole by 2, there will come out $X = x^m + m\times\frac{m-1}{2}x^{m-2}\times\overline{xx-1} + m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}x^{m-4}\times\overline{xx-1}|^2$, &c. and $\sqrt{X^2-1} = \sqrt{xx-1}$ *in*, $mx^{m-1} + m\times\frac{m-1}{2}\times\frac{m-2}{3}x^{m-3}\times\overline{xx-1}$, &c. the former of which being reduced into simple Terms, gives $X = \frac{\overline{2x}|^m}{2} - \frac{m}{2}\times\overline{2x}|^{m-2} + \frac{m}{2}\times\frac{m-3}{2}\times\overline{2x}|^{m-4}$, &c. the very same as above found. And the latter, by multiplying by $\sqrt{-1}$, to

take

take away the imaginary Quantities, and substituting U and u instead of their Equals $\sqrt{1-X^2}$, $\sqrt{1-xx}$, becomes $U = u$ in, $m\times\overline{1-uu}|^{\frac{m-1}{2}} + m\times\frac{m-1}{2}\times\frac{m-2}{3}\times -u^2 \times\overline{1-uu}|^{\frac{m-3}{2}} + m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\overline{1-uu}|^{\frac{m-5}{2}}\times u^4$, &c. which, in like manner, being reduced into simple Terms, will be $U = mu - m\times\frac{m^2-1}{2.3}\times u^3 + m\times\frac{m^2-1}{2.3}\times\frac{m^2-9}{4.5}\times u^5 - m\times\frac{m^2-1}{2.3}\times\frac{m^2-9}{4.5}\times\frac{m^2-25}{6.7}\times u^7$, &c. *Q. E. I.*

COROL. I.

BECAUSE the last Equation, as appears from the Process, will hold as well when m is a Fraction as when a whole Number; let m, or the Multiple Arch A R $(= m\times Ar)$ be supposed indefinitely small; then will $mu - m\times\frac{m^2-1}{2.3}\times u^3 + m\times\frac{m^2-1}{2.3}\times\frac{m^2-9}{4.5}\times u^5$, &c. the Sine of that Arch, or the Arch it self (which in this Case may be considered as equal to it) become $mu + \frac{mu^3}{2.3} + \frac{9mu^5}{2.3.4.5} + \frac{9\times 25u^7}{2.3.4.5.6.7.}$, &c. and therefore the Arch Ar $\left(=\frac{AR}{m}\right)$ whose Sine is u, will, it is manifest, be $= u + \frac{u^3}{2.3} + \frac{3.3u^5}{2.3.4.5} + \frac{3.3.5.5u^7}{2.3.4.5.6.7} + \frac{3.3.5.5.7.7u^9}{2.3.4.5.6.7.8.9}$, &c.

COROL.

COROL. II.

IF Ar be supposed indefinitely small, and m indefinitely great, so that the Multiple Arch $m\times Ar$ $(=A)$ may be a given Quantity; then since u may be considered as equal to Ar, mu will be equal to A, and $mu - m\times\frac{m^2-1}{2.3}\times u^3$, &c. the Sine of A, equal to $mu - \frac{m^3u^3}{2.3} + \frac{m^5u^5}{2.3.4.5}$ or $A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7}$, &c. because 1, 9, 25, &c. in the Factors m^2-1, m^2-9, &c. may here be rejected as indefinitely small in comparison of m^2.

SCHOLIUM.

BECAUSE $\overline{x+\sqrt{xx-1}}|^m + \overline{x-\sqrt{xx-1}}|^m$ is found above to be universally $= \overline{2x}|^m - m\times\overline{2x}|^{m-2} + m\times\frac{m-3}{2}\times\overline{2x}|^{m-4} - m\times\frac{m-4}{2}\times\frac{m-5}{3}\times\overline{2x}|^{m-6}$, &c. it is evident, by Inspection, that $\overline{x+\sqrt{xx+1}}|^m + \overline{x-\sqrt{xx+1}}|^m$ will be $= \overline{2x}|^m + m\times\overline{2x}|^{m-2} + m\times\frac{m-3}{2}\times\overline{2x}|^{m-4}$, &c. and $\therefore$ $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}|^m + \overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}|^m = y^m + my^{m-2}r^2 + m\times\frac{m-3}{2}y^{m-4}r^4$, &c. (by substituting $\frac{y}{2}$ in the room of x, and rr in that of Unity) let r and y be what they will: Therefore, if $y^m + m.y^{m-2}r^2 +$

m

$m\times\frac{m-3}{2}\,y^{m-4}r^4+m\times\frac{m-4}{2}\times\frac{m-5}{3}\,y^{m-6}\,r^6+m\times\frac{m-5}{2}\times\frac{m-6}{3}\times\frac{m-7}{4}\,y^{m-8}r^8$, &c. be supposed equal to some given Quantity c, there will be given $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\big|^{m}+\overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}\big|^{m}$, also $=c$; and therefore $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\big|^{2m}-2r^{2m}+\overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}\big|^{2m}=cc$; wherefore, the double Rectangle of $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\big|^{m}$ into $\overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}\big|^{m}$ being $-2r^{2m}$, the Square of $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\big|^{m}-\overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}\big|^{m}$ will be $=cc+4r^{2m}$, and consequently $\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\big|^{m}-\overline{\frac{y}{2}-\sqrt{\frac{yy}{4}+rr}}\big|^{m}=\sqrt{cc+4r^{2m}}$; which Equation added to the first gives, $2\times\overline{\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}}\big|^{m}=c+\sqrt{cc+4r^{2m}}$; and subtracted therefrom, $2\times\overline{y-\sqrt{\frac{yy}{4}+rr}}\big|^{m}=c-\sqrt{cc+4r^{2m}}$; whence we have $\frac{y}{2}+\sqrt{\frac{yy}{4}+rr}=\overline{\frac{c}{2}+\sqrt{\frac{cc}{4}+r^{2m}}}\big|^{\frac{1}{m}}$, and $\frac{y}{2}-\sqrt{\frac{cc}{4}+r^{2m}}=\overline{\frac{c}{2}-\sqrt{\frac{cc}{4}+r^{2m}}}\big|^{\frac{1}{m}}$ and therefore $y=\overline{\frac{c}{2}+\sqrt{\frac{cc}{4}+r^{2m}}}\big|^{\frac{1}{m}}+\overline{\frac{c}{2}-\sqrt{\frac{cc}{4}+r^{rm}}}\big|^{\frac{1}{m}}$: Which

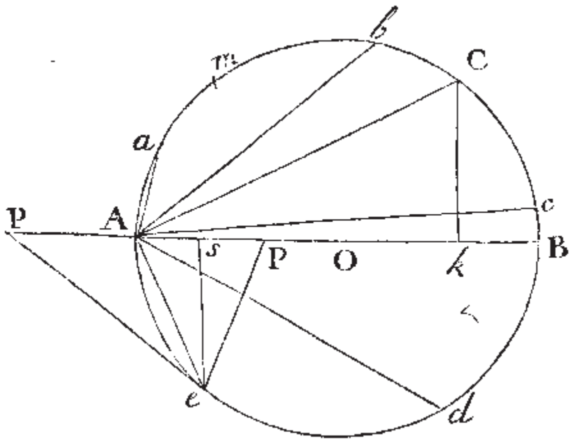
Which may be useful and serve as a Theorem for the Solution of certain Kind of affected Equations, comprehended in this Form, *viz.* $y^m+my^{m-2}r^2+m\times\frac{m-3}{2}\,y^{m-4}r^4$, &c. $=c$: For an Instance hereof, let the cubic Equation $x^3+bx=b$ be proposed; then, by comparing this with $y^m-my^{m-2}r^2$, &c. we have $m=3$, $y=x$, $mr^2=b$, or $rr=\frac{b}{3}$, $c=b$, and consequently $x=\overline{\frac{b}{2}+\sqrt{\frac{bb}{4}+\frac{b^3}{27}}}\big|^{\frac{1}{3}}+\overline{\frac{b}{2}-\sqrt{\frac{bb}{4}+\frac{b^3}{27}}}\big|^{\frac{1}{3}}$

PROPOSITION II.

If on the Diameter AB, *from any Point* C, *in the Circle* ACB, *whose Centre is* O, *the Perpendicular* Ck *be let fall, and the Arc* AC *be divided into any Number,* m, *of equal Parts, as* Aa, am, &c. *and if the whole Periphery be also divided into the same Number of equal Parts, beginning at the Point* a, *as* ab, bc, cd, &c. *and from any Point* P, *in the Diameter* AB, *or* AB *produced, Lines be drawn to the Points* a, b, c, &c. *I say,* $Pa^2\times Pb^2\times Pc^2\times Pd^2$, &c. *the continual Product of the Squares of all those Lines will be equal to* $AO^{2m}\mp AO^{m-1}\times 2Ok\times OP^m+PO^{2m}$.

PUT AO = to 1, PO = to x, AP^2 = to $\overline{1\backsim x}\big|^2\,v$, O$k$ = to b, $2m$ = to n, and the Square of any one of the Chords Aa, Ab, Ac, Ad, &c. equal to z: Then, since any one of the corresponding Arcs Aa, Ab, Abc, &c. reckoned forward a certain Number of Times, brings us to the same Point C, or, is equal to AC, or AC *plus* a certain Number

ber of Times the whole Periphery, it appears from the laſt



Propoſition that $\pm z^{m} \mp n z^{m-1} \pm n \times \frac{n-3}{2} z^{m-2} \mp n \times \frac{n-4}{2} \times \frac{n-5}{3} z^{m-3}$, &c. continued to m Terms, is $= AC^2$, or becauſe AC^2 is $= 2+2b$ $(AB \times Ak)$ it will be $z^{m} - n z^{m-1} + n \times \frac{n-3}{2} z^{m-2} - n \times \frac{n-4}{2} \times \frac{n-5}{3} z^{m-3} \ldots\ldots \mp \overline{2+2b} = 0$, let z ſtand for the Square of which of thoſe Chords you will: Wherefore, the Roots of this Equation being the Squares of the Chords Aa, Ab, Ac, &c. they muſt be all poſitive, their Sum $= n$, the Sum of their Products $n \times \frac{n-3}{2}$, of their Solids $n \times \frac{n-4}{2} \times \frac{n-5}{3}$, &c. by common Algebra. Now, if se be made perpendicular to AB, we ſhall have $AP^2 + Ae^2 \pm AP \times 2As = Pe^2 = AP^2 + Ae^2 \pm AP \times \frac{Ae^2}{AO} = AP^2 + \frac{PO}{AO} \times \overline{Ae}|^2$, which in

in Species, is $Pe^2 = \overline{v+x} \times \overline{Ae}|^2$: And, for the very ſame Reaſons, $Pb^2 = \overline{v+x} \times Ab^2$, $Pc^2 = \overline{v+x} \times Pc^2$, &c. therefore the continual Product of $\overline{v+x} \times Aa^2$ into $\overline{v+x} \times Ab^2$ into $\overline{v+x} \times Ac^2$, &c. is equal to $Pa^2 \times Pb^2 \times Pc^2$, &c. But in the former of theſe Products, it is evident, that when the ſeveral Factors are actually drawn into one another, the Co-efficient of the firſt Term or higheſt Power of v, will be 1; of the next inferior Power, the Sum of all the aboveſaid Roots Aa^2, Ab^2, &c. into x, of the next following, the Sum of all their Products into x^2, &c. and, therefore, the Sum of thoſe Roots being already found $= n$, their Products $= n \times \frac{n-3}{2}$, &c. we have $v^{m} + n x v^{m-1} + n \times \frac{n-3}{2} x^2 v^{m-2} + n \times \frac{n-4}{2} \times \frac{n-5}{3} x^3 v^{m-3} + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} x^4 v^{m-4} \ldots\ldots + \overline{2+2b} \times x^{m} = Pa^2 \times Pb^2 \times Pc^2$, &c. Or, by ſubſtituting for v, its Equal $\overline{1 \backsim x}|^2$ it will be $\overline{1 \backsim x}|^{n} + n x \times \overline{1 \backsim x}|^{n-2} + n \times \frac{n-3}{2} x^2 \times \overline{1 \backsim x}|^{n-4} \ldots\ldots + \overline{2+2b} \times x^{m} = Pa^2 \times Pb^2 \times Pc^2$, &c. (becauſe $2m = n$): This in ſimple Terms is

$$\left.\begin{array}{l}
1 - nx + n \times \frac{n-1}{2} x^2 - n \times \frac{n-1}{2} \times \frac{n-2}{3} x^3, \text{ \&c.} \\
* + nx - n \times \frac{n-2}{1} \times x^2 + n \times \frac{n-2}{1} \times \frac{n-3}{2} x^3, \text{ \&c.} \\
* \quad * + n \times \frac{n-3}{2} \times x^2 - n \times \frac{n-3}{2} \times \frac{n-4}{1} x^3, \text{ \&c.} \\
* \quad * \quad * \quad + n \times \frac{n-4}{2} \times \frac{n-5}{3} x^3, \text{ \&c.} \\
\qquad \text{\&c.} \\
\qquad + \overline{2+b} \times x^{m}
\end{array}\right\} = Pa^2 \times Pb^2, \text{ \&c.}$$

Which

Which contracted, by adding together the homologous Terms, becomes $1 * * *$, *&c.* Hence it appears, that the Coefficients do every where destroy one another, except in the first, last, and the middlemost of the said Terms; and that the middle Term would likewise vanish, if instead of $\overline{2+2b}\times x^m$, the corresponding Term of the above Series $\overline{1 \backsim x}|^n + nx \times \overline{1 \backsim x}|^{n-2}$, or that where the Exponent of x is m, was to be added; wherefore this Term being $n \times \frac{\frac{1}{2}n-1}{2} \times \frac{\frac{1}{2}n-2}{3} \ldots . \frac{1}{\frac{1}{2}n}$ into $x^m (=2x^m)$ as is easy to perceive from the Law of Continuation, we have $1+2bx^m+x^{2m} = Pa^2 \times Pb^2 \times Pc^2$, *&c.* or, $AO^{2m} + 2Ok \times AO^{m-1} \times PO^m + PO^{2m} = Pa^2 \times Pb^2$, *&c.* And, when the Point k is taken on the other Side of O, Ok becoming $-Ok$, $AO^{2m} - 2Ok \times AO^{m-1} \times PO^m + PO^{2m}$ will be equal to $Pa^2 \times Pb^2 \times Pc^2$, *&c.*

Q. E. D.

COROL. I.

IF C be taken at B; then will $Ok = AO$, and $Pa^2 \times Pb^2 \times Pc^2$, *&c.* $= AO^{2m} + 2AO^m \times PO^m + PO^{2m}$; where, by taking the Square Root on each Side, we have $Pa \times Pb \times Pc$, *&c.* $= AO^m + PO^m$.

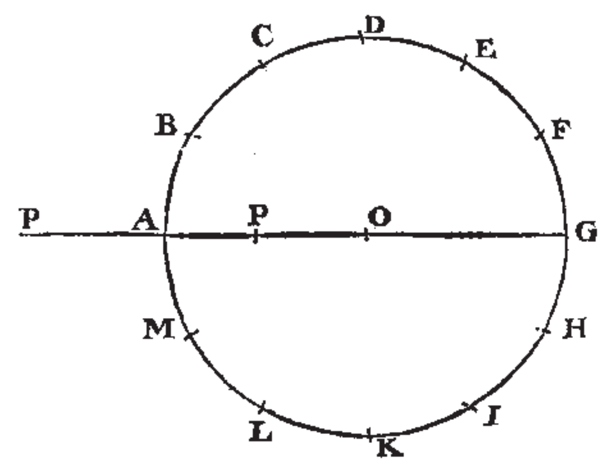
COROL. II.

BUT if C comes into A; then A being $=0$, and $Ok = AO$, $AO^{2m} - 2Ok \times AO^{m-1} \times PO^{2m} = Pa^2 \times Pb^2$, *&c.* will therefore become $AO^{2m} - 2AO^m \times PO^m$

$PO^m + PO^{2m} = Pa^2 \times Pb^2 \times Pc^2$, *&c.* And $Pa \times Pb$, *&c.* $= AO^m \backsim PO^m$.

COROL. III.

HENCE it is manifest, that if any Circle ABCD, *&c.* be divided into as many equal Parts as there are Units in $2m$ (m being any whole Number what-



soever) and if in the Radius OA, produced thro' A, any one of the Points of Division, a Point as P be assumed any where, either within or without the Circle, $PA \times PC \times PE \times PG$, *&c.* will be $= AO^m \backsim PO^m$, $PB \times PD \times PF \times PH$, *&c.* $= AO^m + PO^m$, and $PA \times PB \times PC \times PD \times PE$, *&c.* $= AO^{2m} \backsim PO^{2m}$.